Pseudoforces and general relativity

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Abstract

Pseudoforces appear in classical mechanics while discussing non-inertial reference frames. For instance, in a rotating reference frame, the appearing pseudoforces are the centrifugal force, the Coriolis force and the Euler force. In introductory mechanics courses, they are just treated as conventional forces, but in the context of general relativity, they can also be interpreted as the extrinsic curvature of spacetime.

Introduction

Imagine riding a train while it is accelerating and dropping a rubber ball. From your perspective, the rubber ball will not just bounce up and down on the same spot, but accelerate towards the back of the train - even if we let it go without any horizontal velocity. Of course, we know why this the case - our reference system is accelerated, and it actually is *us* accelerating away from the rubber ball, while it bounces on the same spot viewed from the outside.

Normally, the laws of physics are the same in all reference frames. Coordinates are not physical; they are just something we invented to mathematically describe physical reality. And physics should stay the same no matter what coordinates we draw exactly. This is an extremely powerful principle - for instance, Albert Einstein constructed special relativity just by assuming that the speed of light is the same in all reference frames!

But we hit a limit when trying to use *curved* coordinate systems. Cartesian coordinate systems consist of lines that should be (a) straight, (b) orthogonal to each other and (c) not accelerating. If we break one of these constraints, classical mechanics and special relativity do not work anymore. Consider for instance Newton's force law $\ddot{\vec{x}} = \vec{F}/m$, i. e. $\dot{\vec{x}} = \text{const.}$ when no forces are acting on the object. Such an object would move along a straight line, but if one of the coordinate lines is curved, such a straight line would have $\ddot{\vec{x}} \neq 0$ (and similarly if the coordinate system accelerates over time). Therefore, classical mechanics courses teach that in order to use Newton's laws (or their relativistic version), we need to choose a Cartesian, non-accelerating (and hence non-rotating) coordinate system.

What is gravity?

This is only partially true - differential geometry provides the mathematical framework to construct the relativistic version of Newton's laws in such a way that you can choose arbitrary coordinate systems, including curved or accelerated ones. For instance, in a two-dimensional coordinate system with one time and space coordinate (t, x), we could imagine an accelerated coordinate transformation

$$t' = t \tag{1}$$

$$x' = x - \frac{1}{2}at^2 \tag{2}$$

$$\tag{3}$$

In such an accelerated coordinate system, a nonmoving object object $x(t) = 0$ would look like $x'(t') = -\frac{1}{2}at^2$ - it would be accelerated downwards with acceleration a from the point of view of the accelerated observer, without any interaction with external objects. In the classical, Newtonian picture, we could now define a pseudoforce $F = ma$ acting on any object with mass m such that Newton's second law is still valid.

You may have noticed that this is starting to look a bit like gravity - the gravitational force exerted on an object is also proportional to its mass $F_G = G\frac{Mm}{r^2} \propto m$. Let us picture (x, t) as the coordinate system of an interstellar space station that is non-accelerating, and (x', t') as the coordinate system of a spaceship accelerating away from it with acceleration a. The passengers of this spaceship would naturally use the coordinate system (x', t') to describe their surroundings in the spaceship - with the consequence that they feel a pseudoforce pressing them into their seats. Because this pseudoforce is proportional to the mass of the objects and it is uniform, it would feel remarkably similar to gravity to the passengers. In fact, if the window shutters were down, the passengers would not be able to tell the pseudoforce apart from "real" gravity at all!

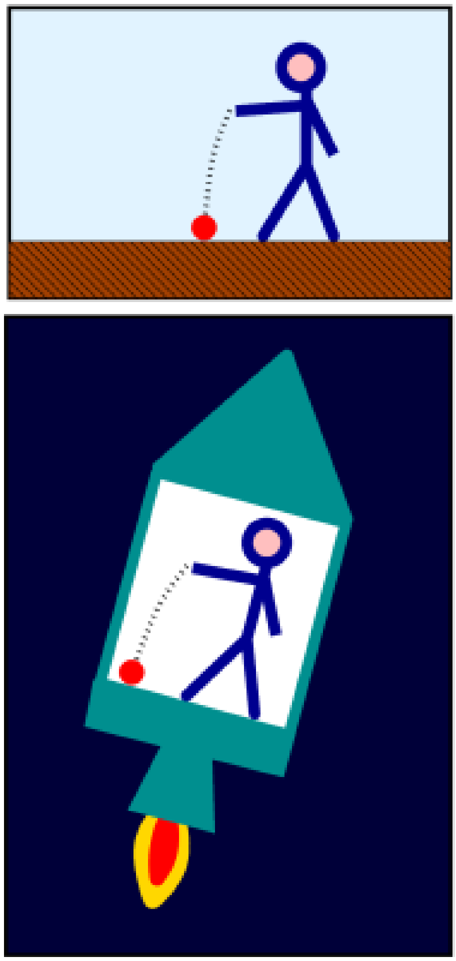

Figure 1: The pseudoforce accelerating objects towards the rear side of the spaceship is indistinguishable from gravity for the passengers. Wikimedia/Mapos

This sort of "artificial gravity" is a common theme in hard science fiction. For instance, in "The Expanse", spaceships excessively accelerate on purpose to generate artificial gravity for their passengers. Another common example is the so-called Stanford torus - a rotating donut-shaped spaceship. Its passengers move along with the rotation, so in order to stay inside the torus, they have to be constantly accelerated towards its center. In the reference frame of the passengers, this means that they constantly experience a pseudoforce dragging them towards the outer wall of the torus. If the rotation speed and the radius of the torus both are sufficiently large, this pseudoforce also will feel like "real" gravity.

After constructing his theory of special relativity, Einstein also did similar thought experiments and formulated what is today known as the "equivalence principle": It's not just possible to tell apart acceleration from gravity - these two things are the same physical phenomenon. Acceleration *is* gravity. When you are sitting in a closed box and see objects being accelerated towards the bottom, there is no experiment you could do to determine whether this acceleration/gravity is caused by a rocket engine or a planet beneath you.

There's a catch to the equivalence principle though - if the box we're in is sufficiently large, we could measure the acceleration at two points far from each other and see whether the acceleration vectors are exactly parallel to each other. If the box is 10 meters wide, the acceleration vectors on both ends would still be perfectly parallel to each other in the "rocket engine" case, but if the box was located on the moon, the angle between them would be about $0.002°$. We see that the equivalence principle only applies *locally* - if the region of spacetime we are observing is infinitesimally small, gravity and external acceleration are the same.

Metric connections and pseudoforces

To measure distances in an arbitrary, non-Cartesian coordinate system, we need a so-called "metric tensor" $g_{\mu\nu}$, which defines the scalar product between two vectors as $\langle v, w\rangle = g_{\mu\nu} v^\mu w^\nu$. Based on the metric tensor, we can define the metric connection[1] ${\Gamma^\mu}_{\nu\rho}$, which tells us how different segments of the coordinate space are linked to each other. We can use this to formulate the so-called *geodesic equation*, which tells us how the tangent vector of a curve has to change with the curve parameter such that the curve is a geodesic, i.e. "straight":

$$\frac{\mathrm{d}^2 x^\mu(\tau)}{\mathrm{d}\tau^2} = -{\Gamma^\mu}_{\nu\rho} \frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \tag{4}$$

where τ is the proper time of the object moving along the curve. So let's try this for the centrifugal force! We're first going to derive the metric of a rotating coordinate system, use it to calculate the relevant Christoffel symbols, substitute it in the geodesic equation and see that once you take the non-relativistic limit, the geodesic equation predicts an acceleration along the r axis.

Deriving the metric

Let's consider a reference frame rotating around the z axis. We'll start by writing down the curvilinear coordinates for a spatial cylindrical coordinate system:

$$x = r\cos\theta \tag{5}$$
$$y = r\sin\theta \tag{6}$$
$$z = z \tag{7}$$

The spacetime interval is given by the pullback of the Euclidean metric along this coordinate transformation:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 && (8)\\ &= (\cos^2\theta + \sin^2\theta)\mathrm{d}r^2 + r^2(\sin^2\theta + \cos^2\theta)\mathrm{d}\theta^2 + \mathrm{d}z^2 && (9)\\ &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\mathrm{d}z^2 && (10) \end{aligned}$$

so we have

$$g_{rr} = 1 \tag{11}$$
$$g_{\theta\theta} = r^2 \tag{12}$$
$$g_{zz} = 1 \tag{13}$$

So far so good, but we could've already guessed that. But it starts getting really interesting once we add in a time coordinate t and set $\theta \to \theta + \omega t$, i.e. make the frame rotating in time:

[1] also known as Levi-Civita connection

$$t = t \tag{14}$$
$$x = r\cos(\varphi) \tag{15}$$
$$y = r\sin(\varphi) \tag{16}$$
$$y = z \tag{17}$$

with $\varphi = \theta + \omega t$.

Now, the line element becomes

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 && (18)\\ &= (1 - r^2\omega^2\cos^2(\varphi) - r^2\omega^2\sin^2(\varphi))\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 + \text{cross terms in t, } \theta && (19)\\ &= (1 - \omega^2 r^2)\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - \mathrm{d}z^2 + \text{cross terms in t, } \theta && (20) \end{aligned}$$

We can directly read off this expression that for high ω, time will pass more slowly for an observer far away from the center of rotation, analogously to the gravitational time dilation effect. This is because regions with higher r rotate faster - indeed, we can also see that at $r = 1/\omega$, $g_{tt} = 0$ as such a point would have to rotate with

Geodesic equation

Let's take the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}$$

At the non-relativistic limit, i.e. $t \approx \tau$, $\mathrm{d}x^\mu/\mathrm{d}\tau \approx (1, 0, 0, 0)$ the geodesic equation for the r coordinate becomes:

$$a := \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\Gamma^r{}_{tt}$$

So we only need to derive a single connection coefficient:

$$\Gamma^r{}_{tt} = \frac{1}{2}g^{r\lambda}\left(2\partial_t g_{t\lambda} - \partial_\lambda g_{tt}\right)$$

The only λ for which $g^{r\lambda} \neq 0$ is $\lambda = r$, but $\partial_t g_{tr} = 0$. So we're left with:

$$\Gamma^r{}_{tt} = -\frac{1}{2}g^{rr}\partial_r g_{tt} = -\omega^2 r$$

Reinserting this back into the geodesic equation, we get:

$$a = r\omega^2$$

or $F = ma = mr\omega^2$, which is precisely the classical result for the centrifugal force.

To summarize, the geodesic equation predicts that in the non-relativistic limit, a geodesic (i.e. straight line in spacetime) will have an acceleration of $r\omega^2$ along the r coordinate in a rotating coordinate system. We can see that the right-hand side of the geodesic equation can be interpreted as the sum of all pseudoforces acting on an object with 4-velocity $\mathrm{d}x^\mu/\mathrm{d}\tau$.

Conclusion

We have seen that the mathematics normally used to describe the curvature of spacetime in general relativity can equally be applied to classical mechanics to derive pseudoforces. In principle, we could now go on to derive e.g. the Coriolis force or the Euler force. This derivation is a nice exercise to see how EInstein's equivalence principle plays out in practice.